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On the problem of multiple M2 branes

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Abstract: A simplified version of $3d$ BL theory is considered, which allows any number N of $M2$ branes in $d = 11$. The underlying 3-algebra structure is provided by degenerate $\mathrm{U}(N)$ Nambu bracket $[X,Y,Z] = \mathrm{tr}(X)\cdot[Y,Z] + \mathrm{tr}(Y)\cdot[Z,X] + \mathrm{tr}(Z)\cdot[X,Y]$, the corresponding f^{abcd} is not totally antisymmetric and the $\mathcal{N} = 8$ supersymmetry of the action remains to be checked. All the fields, including auxiliary non-propagating gauge fields, are in adjoint representation of $\mathrm{SU}(N)$ and the only remnant of 3-algebra structure is an octuple of gauge singlets, acquiring vacuum expectation value in transition to $D2$ branes in $d = 10$.

Keywords: Brane Dynamics in Gauge Theories, D-branes.

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Contents

1. Introduction

In [1] John Schwarz explicitly formulated a problem to find the world-volume $3d$ non-Abelian action with $O\,\mathrm{Sp}(8|4)$ symmetry, including the $\mathcal{N} = 8$ supersymmetry and conformal invariance, which could be used in description of N parallel fundamental $M2$ branes embedded into $11d$ space-time. It was explained in [1] that $3d$ gauge fields in this theory should be non-dynamical and governed by a Chern-Simons action. After important work [2], the problem was finally resolved by Jonathan Bagger and Neil Lambert (BL) in a series of papers [3, 5, 6] and was further elaborated on in [4, 7−11]. The BL construction is based on non-trivial 3-algebra structure and involves a new kind of gauge fields, which do not belong to adjoint representation of naive gauge group $\mathrm{U}(N)$. Moreover, additional antisymmetry requirement imposed in [5] on the 3-algebra structure constants f^{abcd}, leaves only one non-trivial example $f^{abcd} \sim \epsilon^{abcd}$, restricting the choice of the "gauge group" to $\mathrm{U}(2)$ (or, perhaps, $\mathrm{SO}(4)$). A possible way out was actually suggested in [7] and [8], where f^{abcd} were linked to the structure constants f^{abc} of the ordinary gauge group G, say, $G = \mathrm{U}(N)$: $f^{abc0} = f^{abc}$. As already noted at the very end of [7], if one relaxes the unnecessary antisymmetry constraint, this identification essentially implies the use of the standard quantum Nambu 3-bracket for ordinary matrices [12]

$$[X, Y, Z] = \mathrm{tr}(X) \cdot [Y, Z] + \mathrm{tr}(Y) \cdot [Z, X] + \mathrm{tr}(Z) \cdot [X, Y] \tag{1.1}$$

in the role of BL 3-algebra structure.

In what follows we explicitly describe this simplified version of BL construction for arbitrary gauge group G, including $G = \mathrm{U}(N)$ relevant for the stack of N $M2$ branes. The non-trivial BL gauge fields reduce to the pair of ordinary adjoint auxiliary fields, one gauge and one not, very much in the spirit of [8]. We begin in section 2 and section 3 from reminding respectively the $\mathcal{N} = 1$ and $\mathcal{N} = 8$ SUSY non-gauged $3d$ actions from [1] with adjoint octuplet $(\phi^I_{i\bar{j}}, \psi^A_{i\bar{j}})$, $I = 1 \ldots 8$, $A = 1 \ldots 8$, $i, j = 1 \ldots N$. Then in section 4 its simplest gauged generalization is considered, with two auxiliary vector fields: gauge $A^\mu_{i\bar{j}}$ and additional adjoint $B^\mu_{i\bar{j}}$. A non-linear potential *a la* BL is introduced in section 5, involving additional octuplet of gauge singlets (φ^I, χ^A). SUSY invariance of this action is addressed in section 6, where some representative but non-exhaustive examples are presented. In section 7 we briefly remind the main points of original BL construction for $M2$ in $d = 11$ and the way [8] it reduces to $D2$ in $d = 10$. Finally in section 8 we demonstrate that the substitution of (1.1) converts this BL action into the simple formula (5.1) from our section 5, which is the main suggestion of the present paper.

2. Non-gauged $\mathcal{N} = 1$ SUSY action in $3d$

Consider a *real* (Grassmann-valued) $3d$ spinor $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. It exists if space-time signature is $(- + +)$ and three *real*-valued gamma-matrices are $\gamma_0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the Dirac Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{Dir}} = i\bar{\psi}\hat{\partial}\psi &= -\psi^\dagger\sigma_2\Big(- i\sigma_2\partial_0 + \sigma_1\partial_1 + \sigma_3\partial_2\Big)\psi \\
&= i\psi^\dagger\Big(\partial_0 + \sigma_3\partial_1 - \sigma_1\partial_2\Big) \\
&= (\psi_+, \psi_-)\begin{pmatrix} i(\partial_0 + \partial_1) & -i\partial_2 \\ -i\partial_2 & i(\partial_0 - \partial_1) \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \\
&= i\psi_+(\partial_0 + \partial_1)\psi_+ + i\psi_-(\partial_0 - \partial_1)\psi_- - i\psi_+\partial_2\psi_- - i\psi_-\partial_2\psi_+
\end{aligned} \tag{2.1}$$

is non-vanishing, because of the anticommuting nature of ψ-fields. It is related to bosonic Lagrangian

$$\mathcal{L}_{\rm bos} = -\partial^\mu\phi\partial_\mu\phi = (\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 \tag{2.2}$$

by an elementary supersymmetry transformation

$$\begin{aligned}
\delta\phi &= i\bar\psi\varepsilon = (\psi_+,\psi_-)i\sigma_2\begin{pmatrix}\varepsilon_+\\ \varepsilon_-\end{pmatrix} = \psi_+\varepsilon_- - \psi_-\varepsilon_+ = \varepsilon_+\psi_- - \varepsilon_-\psi_+ = i\bar\varepsilon\psi;\\
\delta\psi &= \begin{pmatrix}\delta\psi_+\\ \delta\psi_-\end{pmatrix} = -\hat\partial\phi\varepsilon = \Big(i\sigma_2\partial_0\phi - \sigma_1\partial_1\phi - \sigma_3\partial_2\phi\Big)\begin{pmatrix}\varepsilon_+\\ \varepsilon_-\end{pmatrix}
\end{aligned}\tag{2.3}$$

with constant infinitesimal spinor ε, so that

$$\delta\mathcal{L}_{\rm bos} + \delta\mathcal{L}_{\rm Dir} = \partial_\mu\Big(-i\bar\psi\partial_\mu\phi\varepsilon\Big) \tag{2.4}$$

and the action

$$\int\Big(\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm Dir}\Big)d^3x \tag{2.5}$$

remains invariant.

3. Non-gauged $\mathcal{N} = 8$ SUSY action in 3d

The number of fields can be easily increased: just take M copies of ϕ and ψ. What is non-trivial, for the special value of $M = 8$ the number of supersymmetries can also be increased to M. This can be done by using the special *triality* relation between three different 8-dimensional representations of SO(8) (or, what is essentially the same, triality relation in octonionic algebra). For reasons explained in [9] we separate 3d and 8d gamma-matrices and also denote the scalar fields by ϕ^I rather than X^I.

Fields: scalars ϕ^I and 3d Majorana spinors ψ_A, with I and A labeling components of V_8 and S_8^+ – the vector and spinor representations of SO(8). 4d spinor indices are suppressed.

$\mathcal{N} = 8$ **SUSY transformation**:

$$\begin{aligned}
\delta\phi^I &= i\bar\varepsilon^{\dot A}\Gamma^I_{\dot AA}\psi^A = i\bar\psi^A\Gamma^I_{A\dot A}\varepsilon^{\dot A},\\
\delta\psi_A &= -\hat\partial\phi^I\Gamma^I_{A\dot A}\varepsilon^{\dot A}
\end{aligned}\tag{3.1}$$

Here $\varepsilon = \{\varepsilon^{\dot A}\}$ belongs to the second spinor representation S_8^- of SO(8), the $8\times8\times8$ real-valued structure constants $\Gamma^I_{A\dot A}$ define the *triality* relation between the three 8-dimensional representations of SO(8). They are 8×8 off-diagonal blocks in 16×16 gamma-matrices in 8d:

$$\gamma^I = \begin{pmatrix}0 & \Gamma^I_{A\dot A}\\ \Gamma^I_{\dot AA} & 0\end{pmatrix} = \begin{pmatrix}0 & \Gamma^I\\ \check\Gamma^I & 0\end{pmatrix} \quad \text{and} \quad \gamma^I\gamma^J + \gamma^J\gamma^I = 2\delta^{IJ} \Longrightarrow \Gamma^I\check\Gamma^J + \Gamma^J\check\Gamma^I = 2\delta^{IJ}$$

where $\check{\Gamma}$ denotes transposed matrix. In particular basis $\Gamma^8_{\alpha\beta} = \delta_{\alpha\beta}$ while for $i, j, k = 1 \dots 7$ we have $\Gamma^i_{j8} = -\Gamma^i_{8j} = \delta_{ij}$ and $\Gamma^i_{jk} = c_{ijk}$ where $c_{ijk} = -c_{ikj} = c_{jki}$ are octonionic structure constants, non-vanishing for the following triples:

$$c_{124} = c_{137} = c_{156} = c_{235} = c_{267} = c_{346} = c_{457} = 1 \tag{3.2}$$

Explicitly in this basis

$$\Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad \Gamma^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Gamma^7 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \Gamma^8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Indices I, A and $\dot{A}$ are raised and lowered with the help of invariant metrics g_{IJ}, g_{AB} and $g_{\dot{A}\dot{B}}$.

Invariant action: lagrangian

$$\mathcal{L}_{\text{free}} = -\partial^\mu \phi^I \partial_\mu \phi^I + i\bar{\psi}^A \hat{\partial} \psi^A \tag{3.3}$$

changes under (3.1) by total derivative:

$$\begin{aligned} -2\partial^\mu \phi^I \partial_\mu \Big(i\bar{\psi}^A \Gamma^I_{A\dot{A}} \varepsilon^{\dot{A}} \Big) &\longrightarrow +2i\partial^2 \phi^I \Big(\bar{\psi}^A \Gamma^I_{A\dot{A}} \varepsilon^{\dot{A}} \Big), \\ 2i\bar{\psi}^A \hat{\partial} \Big(-\hat{\partial}\phi^I \Gamma^I_{A\dot{A}} \epsilon^{\dot{A}} \Big) &= -2i\partial^2 \phi^I \Big(\bar{\psi}^A \Gamma^I_{A\dot{A}} \varepsilon^{\dot{A}} \Big) \end{aligned} \tag{3.4}$$

so that the action

$$\int \mathcal{L}_{\text{free}} d^3 x \tag{3.5}$$

remains invariant.

4. Gauged $\mathcal{N} = 8$ SUSY action in $3d$

If ϕ and ψ are promoted to $N \times N$ matrices, or to elements of adjoint representation of any other group G, ϕ^I_a, ψ^A_a, $a = 1 \ldots \dim(G)$, the action acquires a global G-symmetry, which can further be gauged by introduction of the gauge field A^μ_a:

$$\mathcal{L}_{\text{kin}} = -(D^\mu \phi^I)_a (D_\mu \phi^I)_a + i\bar{\psi}^A_a (\hat{D}\psi^A)_a \tag{4.1}$$

with

$$D^{ab}_\mu = \eta^{ab}\partial_\mu + f^{abc}A^c_\mu \tag{4.2}$$

where f^{abc} are the structure constants of G, satisfying Jacobi identity. Indices a are raised and lowered with the help of the Killing metric η_{ab}. However, the action

$$\int \mathcal{L}_{\text{kin}} d^3x \tag{4.3}$$

is not invariant under the $\mathcal{N} = 8$ SUSY transformations

$$\begin{aligned} \delta\phi^I_a &= i\bar{\varepsilon}^{\dot{A}}\Gamma^I_{\dot{A}A}\psi^A_a = i\bar{\psi}^A_a \Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}}, \\ \delta\psi^a_A &= -(\hat{D}\phi_I)^a \Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}} \end{aligned} \tag{4.4}$$

because $(\hat{D}^2 - D^2)_{ab} = \frac{1}{2} f_{abc} F^{\mu\nu}_c \sigma_{\mu\nu} \neq 0$, where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$ and $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] = \epsilon_{\lambda\mu\nu}\gamma^\lambda$. The variation of the action is

$$\delta \int \mathcal{L}_{\text{kin}} d^3x = i\epsilon^{\lambda\mu\nu} \left(\int \bar{\psi}^A_a F^c_{\mu\nu} \gamma^\lambda \phi^I_b d^3x \right) f^{abc} \Gamma^I_{A\dot{A}} \varepsilon^{\dot{A}} \tag{4.5}$$

and can be easily compensated by adding a Chern-Simons-like term

$$S_{\text{CS}} = \epsilon^{\lambda\mu\nu} \int B^a_\lambda F^a_{\mu\nu} d^3x \tag{4.6}$$

with additional auxiliary pseudovector field B^a_μ in adjoint representation of G, which varies under $\mathcal{N} = 8$ SUSY transformation:

$$\delta B^c_\mu = -i\left(\bar{\psi}^A_a \gamma^\lambda \varepsilon^{\dot{A}}\right) \phi^I_b f^{abc} \Gamma^I_{A\dot{A}}, \tag{4.7}$$

while

$$\delta A^c_\mu = 0 \tag{4.8}$$

Note that there is no B^3 term in Chern-Simons action. Therefore one can say that this auxiliary field B works as Lagrange multiplier, nullifying the gauge curvature $F^c_{\mu\nu}$ on-shell, what makes it the theory essentially linear in flat-connection background. Still this theory

$$\int \mathcal{L}_{\text{kin}} d^3x + S_{\text{CS}}, \tag{4.9}$$

though rather trivial, possesses all the desired properties: $\mathcal{N} = 8$ supersymmetry, conformal invariance (at least classical) and it is also P-invariant, provided B^c_μ is a pseudovector.

5. Non linear $\mathrm{U}(N)$ gauged $\mathcal{N}=8$ SUSY action in $3d$

The theory can be made more interesting by introducing non-linear potential and accompanying ψ^2 terms. This can be done for any group G, but we write it in terms of the most interesting $G=\mathrm{U}(N)$, to avoid repetition of formulas from the previous section. Adjoint representation of $\mathrm{SU}(N)$ can be described as anti-Hermitian traceless $N\times N$ matrices, so that the single index $a=1,\ldots,N^2-1$ turns into a pair of indices $a=(i\bar{j})$, $i,j=1\ldots N$, the structure constants f^{abc} are induced by matrix commutators and Killing metric – by matrix trace. Still, formulation in terms of f^{abc} in the previous section is also useful not only for more complicated Lie algebras, it also allows to neglect details, associated with the complex-valuedness of anti-Hermitian matrices. One can also reduce $\mathrm{SU}(N)$ to $\mathrm{SO}(N)$, represented by real antisymmetric $N\times N$ matrices, to fully avoid this kind of problems.

Non-linearization turns to be related to the new $\mathrm{U}(1)$ octuplet (φ^I,χ_A), it can be associated with the unit matrix in $\mathrm{U}(N)$, but in this section we treat these fields simply as gauge singlets, without indices $i\bar{j}$ at all.

	scalars	$\phi^I_{i\bar{j}}$, φ^I	with	$I=1,\ldots,8,$	$i,\bar{j}=1,\ldots,N$
Fields:	spinors	$\psi^A_{i\bar{j}}$, χ_A	with	$A=1,\ldots,8,$	$i,\bar{j}=1,\ldots,N$
	vectors	$A^\mu_{i\bar{j}}$, $B^\mu_{i\bar{j}}$	with	$\mu=0,1,2,3,$	$i,\bar{j}=1,\ldots,N$

Lagrangian:

$$\begin{aligned}
&-\mathrm{tr}(\mathcal{D}_\mu\phi^I)^2+i\,\mathrm{tr}\,\bar{\psi}^A\hat{\mathcal{D}}\psi^A+\epsilon^{\lambda\mu\nu}\mathrm{tr}\;F_{\mu\nu}B_\lambda-\\
&\quad-(\partial_\mu\varphi^I)^2+i\bar{\chi}^I\hat{\partial}\chi^I+2i\varphi^I\mathrm{tr}\Big(\phi^J[\bar{\psi}^A,\psi^B]\Big)\Gamma^{IJ}_{AB}\\
&\qquad+i\,\mathrm{tr}\Big([\phi^I,\phi^J]\bar{\psi}^A\Big)\chi^B\Gamma^{IJ}_{AB}+\sum_{K\neq I,J}(\varphi^K)^2\mathrm{tr}\left([\phi^I,\phi^J]\right)^2
\end{aligned}\tag{5.1}$$

Sums are taken over repeated indices of all kinds. The long derivatives here are

$$\begin{aligned}
(\mathcal{D}^\mu\phi)^I_{i\bar{j}}&=\partial^\mu\phi^I_{i\bar{j}}+[A^\mu,\phi^I]_{i\bar{j}}+B^\mu_{i\bar{j}}\varphi^I,\\
(\mathcal{D}^\mu\psi)^A_{i\bar{j}}&=\partial^\mu\psi^A_{i\bar{j}}+[A^\mu,\psi^A]_{i\bar{j}}+2B^\mu_{i\bar{j}}\chi^A
\end{aligned}\tag{5.2}$$

$N=8$ SUSY transformations:

$$\begin{aligned}
\delta\varphi^I&=i\bar{\varepsilon}^{\dot{A}}\Gamma^I_{A\dot{A}}\chi^A,\\
\delta\chi_A&=-\hat{\partial}\varphi^I\Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}},\\
\delta\phi^I_{i\bar{j}}&=i\varepsilon^{\dot{A}}\Gamma^I_{A\dot{A}}\psi^A_{i\bar{j}},\\
(\delta\psi_A)_{i\bar{j}}&=-(\hat{\mathcal{D}}\phi^I)_{i\bar{j}}\Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}}-[\phi^I,\phi^J]_{i\bar{j}}\varphi^K\Gamma^{IJK}_{A\dot{A}}\varepsilon^{\dot{A}},\\
\delta A^\lambda_{i\bar{j}}&=\left(\varphi^I\bar{\psi}^A_{i\bar{j}}+\phi^I_{i\bar{j}}\bar{\chi}^A\right)\gamma^\lambda\Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}},\\
\delta B^\lambda_{i\bar{j}}&=[\bar{\psi}^A,\phi^I]_{i\bar{j}}\gamma^\lambda\Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}}
\end{aligned}\tag{5.3}$$

The "linear" action (4.9) — the first line in (5.1) — and the SUSY transformations (4.4), (4.7), (4.8) from the previous section are reproduced if we put all $\varphi^I = \chi^A = 0$, what is a consistent *reduction* of the theory.

Instead one can ascribe an average value to the gauge singlet φ, say

$$\left\langle \varphi^{I=8} \right\rangle = g_{\rm YM} \tag{5.4}$$

then the B field acquires a "mass term" from $\mathrm{tr}\Big((D\phi)^8\Big)^2 \to g_{\rm YM}^2 \mathrm{tr}(B_\mu)^2$ and integrating B out one obtains a kinetic term $\frac{1}{g_{\rm YM}^2}\mathrm{tr}F_\mu\nu^2$ for the gauge field. This is the $M2 \to D2$ transition, described in [8] for the general BL theory. Of course, eq. (5.4) is inconsistent with equations of motion and is not an allowed VEV in the theory (5.1). One can cure this by adding non-trivial potential to the field φ, what necessarily breaks a part of supersymmetry as well as conformal invariance — as required in transition to $D2$ branes.

6. On SUSY invariance of (5.1)

Detailed check of invariance of (5.1) is somewhat sophisticated and remains beyond the scope of the present paper, we give in this section only some illustrative examples. The real problem is of course the lack of superfield or any other explicitly supersymmetric formulation. Non-linear terms in BL action are not expressed even through an $\mathcal{N} = 1$ superpotential, i.e. do not look like $\left(\frac{\partial W}{\partial \phi^I}\right)^2 + \bar\psi^I \frac{\partial^2 W}{\partial\phi^I\partial\phi^J}\psi^J$ for some $W(\phi)$, though they look surprisingly close to this for a theory with highly extended supersymmetry – what can serve as additional inspiration for the search of invariant formulations. We remind the "natural" logic of invariance check for BL action in the next section 7, while here we directly collect the terms of *some* different structures in the variation of (5.1). Of primary importance is variation of the potential, since it has the open dependence on the matric h_{ab} and thus can cause problems for non-antisymmetric f^{abcd}.

6.1 $\varphi\phi^4\chi$ terms

These terms come from the variation of φ in the potential and from non-linear variation of ψ in the last fermionic item of (5.1):

$$\sum_{K\neq I,J} 2i(\bar\varepsilon\Gamma^K\chi)\varphi^K \mathrm{tr}\left([\phi^I,\phi^J]\right)^2 + i\mathrm{tr}\Big([\phi^I,\phi^J][\phi^L,\phi^M]\Big)(\bar\varepsilon\Gamma^{KLM}\Gamma^{IJ}\chi)\varphi^K =$$
$$= i\sum_{K\neq I,J}\varphi^K \mathrm{tr}\left\{[\phi^I,\phi^J]\Big(2[\phi^I,\phi^J](\bar\varepsilon\Gamma^K\chi) + [\phi^L,\phi^M](\bar\varepsilon\Gamma^{IJK}\Gamma^{LM}\chi)\Big)\right\} \tag{6.1}$$

How can these very different structures cancel each other? The reason is that antisymmetry in KLM and in IJ allows one to arbitrarily change the order of Γ-matrices in Γ^{KLM} and Γ^{IJ} and then Jacobi identity for the trace over ϕ-fields can be applied to ensure the cancelation.

Let $K = 8$. Then there are three different kinds of terms in remaining sum over $IJLM$.

- The pair LM coincides with IJ, say, $I,J=1,2$ and either $L,M=1,2$ or $L,M=2,1$. These terms contribute:

$$2[\phi^1,\phi^2]\Gamma^8+[\phi^1,\phi^2]\Gamma^{812}\Gamma^{12}+[\phi^2,\phi^1]\Gamma^{812}\Gamma^{21}=0$$

because $\Gamma^{812}\Gamma^{12}=(-\Gamma^8\Gamma^2\Gamma^1)(\Gamma^1\Gamma^2)=-\Gamma^8$, while $\Gamma^{812}\Gamma^{21}=+\Gamma^8$.

- Both indices L and M is different from I and J, say $I,J=1,2$ and $L,M=3,4$. The corresponding contribution is obtained by reordering commutators under the sign of trace:

$$\mathrm{tr}\left\{\phi^4\left(\Big[\phi^3,[\phi^1,\phi^2]\Big]\Gamma^{812}\Gamma^{43}+\Big[\phi^2,[\phi^1,\phi^3]\Big]\Gamma^{813}\Gamma^{42}+\Big[\phi^1,[\phi^2,\phi^3]\Big]\Gamma^{823}\Gamma^{41}\right)\right\}=0$$

due to Jacobi identity for the commutators, which can be applied because all Γ-matrix structures are the same: proportional to $\Gamma^{84}\Gamma^{123}$. Note that one of the indices L,M (but not I,J!) could be equal to $K=8$.

- One of the indices LM coincides with one of IJ, another — not, say $I,J=1,2$ and $L,M=1,3$ or $I,J=1,3$ and $L,M=1,2$. Such terms contribute

$$\begin{aligned}&\mathrm{tr}\Big([\phi^1,\phi^2][\phi^1,\phi^3]\Big)\Gamma^{812}\Gamma^{13}+\mathrm{tr}\Big([\phi^1,\phi^3][\phi^1,\phi^2]\Big)\Gamma^{813}\Gamma^{12}\\&\qquad=-\mathrm{tr}\Big([\phi^1,\phi^2][\phi^1,\phi^3]\Big)\Gamma^{81}\Gamma^1\Big(\Gamma^3\Gamma^2+\Gamma^2\Gamma^3\Big)=0\end{aligned}\tag{6.2}$$

because of anti-commutativity of Γ-matrices. Once again, one of the indices L,M could be $K=8$.

It is important here that the terms with $K=I$ or J are excluded from the potential, because such terms would contribute to the first item in (6.1), but not to the second one (where L and M are different from K) thus no cancelation occurs and supersymmetry would be broken.

6.2 $\varphi^2\phi^3\psi$ terms

The story about these terms is very similar: they come from the variation of ϕ in the potential and from non-linear variation of ψ in the first bi-fermionic item in the second line of (5.1):

$$\sum_{K\neq I,J}4i(\varphi^K)^2\mathrm{tr}\Big([\phi^I,\phi^J][\phi^I,\bar\psi^A]\Big)\Gamma^J_{A\dot A}\varepsilon^{\dot A}+4i\varphi^I\mathrm{tr}\Big([\phi^J,\bar\psi^A][\phi^K,\phi^L]\Big)\varphi^M\Gamma^{IJ}_{AB}\Gamma^{KLM}_{B\dot B}\varepsilon^{\dot B}\tag{6.3}$$

The second term can be rewritten as

$$\varphi^I\varphi^M\mathrm{tr}\left(\bar\psi^A\Big[\phi^J,[\phi^K,\phi^L]\Big]\right)\Gamma^{IJ}_{AB}\Gamma^{KLM}_{B\dot B}\varepsilon^{\dot B}$$

Now we take into account Jacobi identity for commutators and $I\leftrightarrow M$ symmetry.

- Let first $I \neq M$, say, $I, M = 1, 2$. If all $J, K, L \neq 1, 2$ then $\Gamma^{1J}\Gamma^{KL2} + \Gamma^{2J}\Gamma^{KL1} = 0$. If at least two of the three indices J, K, L coincide with I or M, one of the two Γ-factors vanishes. The only interesting case is when one of the three coincides with I or M, say, $J, K, L = 1, 3, 4$ Then we get:

$$\begin{aligned}
&\varphi^1\varphi^2\Big\{\Big[\phi^1,[\phi^3,\phi^4]\Big]\Big(\Gamma^{11}\Gamma^{342}+\Gamma^{21}\Gamma^{341}\Big)+\Big[\phi^3,[\phi^1,\phi^4]\Big]\Big(\Gamma^{13}\Gamma^{142}+\Gamma^{23}\Gamma^{141}\Big) \\
&\quad +\Big[\phi^4,[\phi^1,\phi^3]\Big]\Big(\Gamma^{14}\Gamma^{132}+\Gamma^{24}\Gamma^{131}\Big)\Big\} = \\
&\quad\quad = \varphi^1\varphi^2\left\{\Big[\phi^1,[\phi^3,\phi^4]\Big]-\Big[\phi^3,[\phi^1,\phi^4]\Big]+\Big[\phi^4,[\phi^1,\phi^3]\Big]\right\}\Gamma^{234}=0
\end{aligned} \tag{6.4}$$

because of Jacobi identity.

- Let now $I = M$. Then in the second term in (6.3) all $J, K, L \neq I$, and it is equal to

$$\sum_{I\neq J,K,L} (\varphi^I)^2 \mathrm{tr}\left(\bar\psi^A\Big[\phi^J,[\phi^K,\phi^L]\Big]\right)(\Gamma^J\Gamma^{KL})_{A\dot A}\varepsilon^{\dot A} \tag{6.5}$$

If all the three indices J, K, L are different, this sum vanishes dues to Jacobi identity; if all the three coincide than Γ^{KL} is identical zero. The only interesting cases are $J = K$ and $J = L$, when (6.5) exactly cancels the first term in (6.3). Note that again it is important that terms with $K = I, J$ are excluded from the potential in (5.1).

6.3 Terms with DB

Such terms appear from variation of kinetic part of the action in the first line of (5.1) and are compensated by the variation of A-field in the Chern-Simons action: both are proportional to the long derivative

$$\epsilon_{\lambda\mu\nu}(D^\mu B^\nu)_{i\bar j} = \epsilon_{\lambda\mu\nu}\Big(\partial^\mu B^\nu_{i\bar j} + [A^\mu, B^\nu]_{i\bar j}\Big)$$

Note once again that there are no B^3 terms in (5.1), there is nothing to cancel their variation and they would violate supersymmetry. At the same time the $\varphi^2 B^2$ term is present in the first line of (5.1): as we discussed, it plays the role in transition to $D2$ branes, but its variation cancels against that of the $\psi B\chi$ and does not produce B^2 terms. Only DB is present in the variation.

In a little more detail, the first line in (5.1) can be rewritten as

$$-\mathrm{tr}(D_\mu\phi^I)^2 - 2\varphi^I\mathrm{tr}(B^\mu D_\mu\phi^I) - (\varphi^I)^2\mathrm{tr}\, B_\mu^2 + i\,\mathrm{tr}\bar\psi^A\hat D\psi^A + 2i\,\mathrm{tr}(\bar\psi^A\hat B)\chi^A \tag{6.6}$$

where $D_\mu\phi = \partial_\mu\phi + [A_\mu, \phi]$ is an ordinary adjoint long derivative with the gauge field A_μ. The B^2 terms in the SUSY variation are

$$-2i\varphi^I(\bar\varepsilon\Gamma^I\chi)\mathrm{tr}\, B_\mu^2 + 2i\,\mathrm{tr}(\bar\varepsilon\Gamma^I\hat B)\hat B\chi = 0$$

because $\mathrm{tr}\,\hat B^2 = \mathrm{tr}\, B^2$. Note that coefficient 2 in front of the last terms in (6.6) — and thus in the second long derivative in (5.2), — as well as anti-Hermiticity (antisymmetry) of B are important for this cancelation.

Similarly the variation of other terms in (6.6) provide terms with $F = \partial A + [A, A]$ and $DB = \partial B + [A, B]$ canceled respectively by the variations of B and A in the Chern-Simons term: $\delta\int \mathrm{tr}FB = \int \mathrm{tr}(F\delta B - DB\delta A)$.

6.4 Other terms

The terms which are cubic in fermion fields, $\varphi\chi\psi^2$ and $\varphi\psi^3$, come from the variation of gauge fields in the first line of (5.1) and from variation of scalars in bi-fermion terms in the second line. Their cancelation requires adjustement of the relative coefficient between the first and the second lines of (5.1) and depends on the Fierz identities for gamma-matrices. The latter can be handled by making use of explicit Γ-matrix representation from section 3. Most numerous are terms of the type $\varphi\phi^2\psi$, they come from many places in (5.1) and we do not analyze them in this paper. Only when such analysis is completed one can take the action (5.1) really serious.

7. BL construction

In the remaining part of this paper we comment on the place of (5.1) in the general BL theory. As already mentioned in the introduction, (5.1) is associated with the special version (1.1) of the 3-bracket — the only one known for generic group $\mathrm{U}(N)$. It does *not* satisfy the antisymmetry requirement for f^{abcd} and original $SO(4) = SU(2) \times SU(2)$ example of [5] with $f^{abcd} = \epsilon^{abcd}$ is *not* the same as (5.1) for $N = 2$. The SO(4) example is associated with the deformation of (1.1) by addition of $I \cdot \mathrm{tr}(A[B, C])$, with the unit matrix $I \in \mathrm{U}(N)$, which exists — at least in such simple form — only for $N = 2$.

7.1 Action and its invariance

This section is a very brief summary of BL construction, based on arbitrary (axiomatically defined) 3-product. The story begins at the end of section 3. Note that we use the same notations a, b, c, d and "tr" as in ssection 4, 5, but here they correspond to a somewhat different group $\tilde{G} \neq G$: the simplified BL action (5.1) with the gauge group $G = SU(N)$ is associated with $\tilde{G} = \mathrm{U}(N)$.

7.1.1 Kinetic terms

Let ϕ and ψ belong to some *real* representation R of some gauge group $\tilde{G}$ with connection $\tilde{\mathcal{A}}$. Then short derivatives in (3.3) are substituted by the long ones, and the $\tilde{G}$-invariant scalar product is involved (denoted by "tr"):

$$\mathcal{L}_{\mathrm{kin}} = -\mathrm{tr}_R\, \tilde{\mathcal{D}}^\mu\phi^I\tilde{\mathcal{D}}_\mu\phi^I + i\mathrm{tr}_R\, \bar{\psi}^A\hat{\tilde{\mathcal{D}}}\psi^A \tag{7.1}$$

Under the same transformation (3.1) we get:

$$\delta S_{\mathrm{kin}} = \int \delta\mathcal{L}_{\mathrm{kin}}d^3x = i\int \mathrm{tr}_R\, \bar{\psi}^A\tilde{\mathcal{F}}_{\mu\nu}\sigma^{\mu\nu}\phi^I\Gamma^I_{A\dot{A}}\varepsilon^{\dot{A}}, \tag{7.2}$$

(of course, $\varepsilon^{\dot{A}}$, is also an element of R).

7.1.2 Chern-Simons self-interaction of the gauge field

In order to compensate for this change one can add to $S_{\rm kin}$ a Chern-Simons action for $\tilde{\mathcal{A}}$,

$$S_{\rm CS} = {\rm Tr}\int \Big(\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \frac{2}{3}\tilde{\mathcal{A}}^3\Big) = \epsilon^{\lambda\mu\nu}{\rm Tr}\int \Big(\tilde{\mathcal{A}}_\lambda\partial_\mu\tilde{\mathcal{A}}_\nu + \frac{2}{3}\tilde{\mathcal{A}}_\lambda\tilde{\mathcal{A}}_\mu\tilde{\mathcal{A}}_\nu\Big)d^3x \tag{7.3}$$

which varies as

$$\delta S_{\rm CS} = \epsilon^{\lambda\mu\nu}{\rm Tr}\int \tilde{\mathcal{F}}_{\mu\nu}\delta\tilde{\mathcal{A}}_\lambda = \epsilon^{\lambda\mu\nu}\int \tilde{\mathcal{F}}^{ab}_{\mu\nu}\delta\tilde{\mathcal{A}}^{ab}_\lambda \tag{7.4}$$

and can compensate the change in (7.2) if (3.1) is complemented by

$$\delta\tilde{\mathcal{A}}_\lambda = -i\phi^I\otimes\bar\psi^A\gamma_\lambda\Gamma^I_{A\dot A}\varepsilon^{\dot A} \quad \text{or} \quad \delta\tilde{\mathcal{A}}^\lambda_{ab} = -i\bar\psi^A_a\gamma^\lambda\Gamma^I_{A\dot A}\varepsilon^{\dot A}\phi^I_b \tag{7.5}$$

Note that "Tr" is different from "tr": while the latter one is for representation R where ϕ, ψ and ε belong, the former one is over representation $R\otimes R$, where connection $\tilde{\mathcal{A}}$ is taking its values.

7.1.3 Twisted Chern-Simons

The same result can be achieved for any action, which changes by

$$\delta\tilde{S} = \epsilon^{\lambda\mu\nu}{\rm Tr}\int \tilde{\mathcal{F}}_{\mu\nu}\delta\mathcal{A}_\lambda \tag{7.6}$$

provided

$$\delta\mathcal{A}_\lambda = -i\phi^I\otimes\bar\psi^A\gamma_\lambda\Gamma^I_{A\dot A}\varepsilon^{\dot A} \quad \text{or} \quad \delta\mathcal{A}^\lambda_{ab} = -i\bar\psi^A_a\gamma^\lambda\Gamma^I_{A\dot A}\varepsilon^{\dot A}\phi^I_b \tag{7.7}$$

An example is provided by "twisted Chern-Simons" action

$$\tilde{S}_{\rm CS} = {\rm Tr}\int \Big(\mathcal{A}d\tilde{\mathcal{A}} + \frac{2}{3}\mathcal{A}\tilde{\mathcal{A}}^2\Big) = \epsilon^{\lambda\mu\nu}{\rm Tr}\int \Big(\mathcal{A}_\lambda\partial_\mu\tilde{\mathcal{A}}_\nu + \frac{2}{3}\mathcal{A}_\lambda\tilde{\mathcal{A}}_\mu\tilde{\mathcal{A}}_\nu\Big)d^3x \tag{7.8}$$

with

$$\tilde{\mathcal{A}}^{ab} = f^{abcd}\mathcal{A}_{cd} \tag{7.9}$$

7.1.4 Interaction terms

Non-trivial transformation (7.5) of the $\tilde{\mathcal{A}}$-field contributes new terms to (7.2):

$${\rm Tr}\int \delta\tilde{\mathcal{A}}_\mu\Big(2\phi^I\otimes\mathcal{D}^\mu\phi^I - i\bar\psi^A\otimes\gamma^\mu\psi^A\Big)d^3x \tag{7.10}$$

These can be compensated by simultaneous addition of interaction terms to the action non-linear terms to the SUSY transformation (3.1). As shown in [5] this can be done if (7.7) is used instead of (7.5), and all additions are expressed in terms of the "structure constant" f^{abcd} from (7.9), which can be used to define a 3-product

$$R\otimes R\otimes R\to R: \quad [X,Y,Z]^a = f^{abcd}X_bY_cZ_d \tag{7.11}$$

To raise and lower indices one also needs a metric h_{ab}, which, however, does not show up in the supersymmetry transformations. In these terms

$$\mathcal{L}_{\rm int} = \frac{1}{6}{\rm tr}_R[\phi^I,\phi^J,\phi^K]^2 + \frac{1}{2}{\rm tr}_R\ \bar\psi^A\Gamma^{IJ}_{AB}[\phi^I,\phi^J,\psi^B] \tag{7.12}$$

and

$$\delta\psi_A = \hat{\phi}^I\Gamma^I_{A\dot A}\varepsilon^{\dot A} + \frac{1}{6}[\phi^I,\phi^J,\phi^K]\Gamma^{IJK}_{A\dot A}\varepsilon^{\dot A} \tag{7.13}$$

The full action [5]

$$\int \mathcal{L}_{\rm kin}d^3x + \tilde S_{\rm CS} + \int \mathcal{L}_{\rm int}d^3x \tag{7.14}$$

is invariant, provided the 3-product satisfies the Jacobi-like "fundamental identity" [13, 4–6]

$$\Big[A,B,[C,D,E]\Big] = \Big[[A,B,C],D,E\Big] + \Big[C,[A,B,D],E\Big] + \Big[C,D,[A,B,E]\Big] \tag{7.15}$$

and f^{abcd} has certain symmetry properties. It is usually required to be totally antisymmetric, though this requirement can probably be relaxed [7].

7.1.5 Summary of transformations

$$\begin{aligned}
&\delta\Big((\tilde{\mathcal{D}}\phi^I)^2 + \bar\psi^A_a(\hat{\tilde{\mathcal{D}}}\psi^A)_b + \tilde S_{\rm CS}\Big)\\
&\qquad \delta\tilde{\mathcal{A}}^{ab}_\mu\left(\phi^I_a(\tilde{\mathcal{D}}_\mu\phi^I)_b + \bar\psi^A_a\gamma_\mu\psi^A_b\right) + \bar\psi^A_a\frac{\partial H^a_{A\dot A}}{\partial\phi^I_b}(\hat{\tilde{\mathcal{D}}}\phi^I)_b\varepsilon^{\dot A}\\
&\delta\Big(\bar\psi^A_a\psi^B_b\Gamma^{IJ}_{AB}T^{IJ}_{ab}(\phi)\Big)\\
&\qquad +\bar\psi^A_a\psi^B_b\Gamma^{IJ}_{AB}\frac{\partial T^{IJ}_{ab}}{\partial\phi^K_c}\bar\psi^C_c\Gamma^K_{C\dot C}\varepsilon^{\dot C} + 2\bar\psi^A_a\Gamma^{IJ}_{AB}T^{IJ}_{ab}\Gamma^K_{B\dot B}(\hat{\tilde{\mathcal{D}}}\phi^K)_b\varepsilon^{\dot B} + 2\bar\psi^A_a\Gamma^{IJ}_{AB}T^{IJ}_{ab}H^b_{B\dot B}\varepsilon^{\dot B}\\
&\delta\Big(V(\phi)\Big)\\
&\qquad +\frac{\partial V}{\partial\phi^I_a}\bar\psi^A_a\Gamma^I_{A\dot A}\varepsilon^{\dot A}
\end{aligned}$$

There are three terms in the first line, three in the second and one in the forth. Enumerating them, from 1 to 7, we have the following cancelations to take place:

$$\begin{array}{lll}
2+4 & \delta\tilde{\mathcal{A}}^{ab}_\mu\bar\psi^A_a\gamma_\mu\psi^A_b + \bar\psi^A_a\psi^B_b\Gamma^{IJ}_{AB}\dfrac{\partial T^{IJ}_{ab}}{\partial\phi^K_c}\bar\psi^C_c\Gamma^K_{C\dot C}\varepsilon^{\dot C} & =0\\
6+7 & 2\bar\psi^A_a\Gamma^{IJ}_{AB}T^{IJ}_{ab}H^b_{B\dot B}\varepsilon^{\dot B} + \dfrac{\partial V}{\partial\phi^I_a}\bar\psi^A_a\Gamma^I_{A\dot A}\varepsilon^{\dot A} & =0\\
1+3+5 & \delta\tilde{\mathcal{A}}^{ab}_\mu\phi^I_a(\tilde{\mathcal{D}}_\mu\phi^I)_b + \bar\psi^A_a\dfrac{\partial H^a_{A\dot A}}{\partial\phi^I_b}(\hat{\tilde{\mathcal{D}}}\phi^I)_b\varepsilon^{\dot A} + 2\bar\psi^A_a\Gamma^{IJ}_{AB}T^{IJ}_{ab}\Gamma^K_{B\dot B}(\hat{\tilde{\mathcal{D}}}\phi^K)_b\varepsilon^{\dot B} & =0
\end{array}$$

Direct check of these cancelations involves application of Fierz and other γ-matrix identities and is rather tedious.

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7.2 From $M2$ to $D2$

A very important and instructive transition from BL action for $M2$ branes in $d = 11$ with 8 transverse fields ϕ^I to $D2$ branes in $d = 10$ with only 7 transverse fields ϕ^i an the action

$$\int d^3x \left(\sum_{i=1}^{7} (D_\mu \phi^i)_a (D^\mu \phi^i)_a + \frac{1}{4g_{\rm YM}^2} \operatorname{tr} F_a^{\mu\nu} F_{\mu\nu}^a + \text{fermions} \right) \tag{7.16}$$

was considered in [8]. Note that both ϕ^a and A_μ^a are now in the same adjoint representation of the gauge group.

As explained in [8], transition from $M2$ action to $D2$ one in (7.16) takes place when one of the scalar fields, associated with the 8-th transverse direction, acquires vacuum expectation value:

$$< \phi_{a=0}^{I=8} >= g_{\rm YM} \tag{7.17}$$

and a in (7.16) takes one less value than a in BL action (7.14), so that $\dim(\tilde{G}) = \dim(G)+1$. We label this extra value of a in $\tilde{G}$ by 0 (note that it is *not* the additional 0-generator of [5]!). Then for $a \neq 0$

$$\begin{aligned} (\tilde{\mathcal{D}}_\mu \phi^I)_a = \partial_\mu \phi_a^I - \tilde{\mathcal{A}}_\mu^{ab} \phi_b^I &= \partial_\mu \phi_a^I - f_{abcd} \mathcal{A}_\mu^{cd} \phi_b^I \\ &= \partial_\mu \phi_a^I - f_{abc0} \mathcal{A}_\mu^{c0} \phi_b^I - f_{a0cd} \mathcal{A}_\mu^{cd} \phi_0^I \\ &= \partial_\mu \phi_a^I - f_{abc} A_\mu^c \phi_b^I - B_\mu^a \phi_0^I \end{aligned} \tag{7.18}$$

where

$$f^{abc} = f^{abc0} \tag{7.19}$$

are the ordinary structure constants of G.

The terms with $I \neq 8$ in (7.16) provide

$$\sum_{i=1}^{7} \left(\partial_\mu \phi_a^i + f_{abc} A_\mu^c \phi_b^i \right)^2 + O(B\phi^2) = \sum_{i=1}^{7} \Big((D_\mu \phi^i)_a \Big)^2 + O(B\phi^2) \tag{7.20}$$

to the action, while those with $I = 8$ contribute

$$g_{\rm YM}^2 B_\mu^a B_a^\mu \tag{7.21}$$

In combination with

$$S_{\rm CS} = \int (BF + B^3) \tag{7.22}$$

it provides the kinetic term $\frac{1}{4g_{\rm YM}^2} \int \operatorname{tr} F^2 d^3x$ for Yang-Mills field.

Note that the *number* of B-fields

$$B_\mu^a = f_{a0cd} A_\mu^{cd} \tag{7.23}$$

is smaller than that of A_μ^{cd}, so that in the quantum case the measure $[\mathcal{D}A]$ in functional integral would differ from $[\mathcal{D}B]$ by non-trivial Jacobian factor.

7.3 3-algebra and associated gauge fields

The key point of BL construction is the very interesting 3-algebra structure, supposedly generalizing the Lie-algebra structure underlying the ordinary gauge symmetry. BL construction associates with 3-algebras a new kind of gauge fields A_μ^{ab}. If the ϕ^I-fields are interpreted as X^I, describing 8 transverse directions to the $M2$ brane world volume in embedding $d = 11$ space time, then index a is naturally treated as $a = (i\bar{j})$ where i and $\bar{j}$ label the copies of $M2$ brane and associated string stretched between them. Despite this interpretation is natural for $D2$ and not obligatory to $M2$ branes, the reduction procedure [8], briefly described in the previous section 7.2, strongly suggests that it also holds for $M2$. Then A^{ab} is actually carrying two pairs of indices $A^{ab} = A^{i\bar{j},k\bar{l}} = A^{ij}_{kl}$ and looks like associated with a *pair of strings*, i.e. with a (fundamental) 2-brane stretching between the two strings which stretch between the two pairs of $M2$ branes. This is of course more than natural for the M-theory and this is what makes the BL construction so attractive. Unfortunately, *this* intriguing structure is fully suppressed in the simplified action (5.1), where gauge fields are the ordinary adjoints, associated with pairs of branes rather than pairs of strings between them.

The big problem of BL construction is that no fully-non-trivial examples of 3-algebras are known. Spectacular original SO(4) example of [5] appeared to be difficult to generalize, despite certain effort in [4]–[11]. Immediate idea about octonionic generalizations was shown in [9] to be not so straightforward. Even the very inspiring reformulations of [4, 5, 7] do not provide immediate new examples. One could hope that such examples are *obliged* to exist, because N branes with arbitrary N certainly exist. Unfortunately, eq. (5.1) demonstrates that these configurations can probably be described by a very primitive BL action, which is associated with degenerate 3-bracket (1.1), as we shall see in the next section.

Before proceeding to this last subject of the present paper, it deserves emphasizing that the total-antisymmetricity requirement imposed in [5] over-constrains the possible 3-algebra structures. The 3-bracket (1.1) corresponds to f^{abcd} which is *not* totally antisymmetric, in particular, $f^{0bcd} = 0$ while $f^{abc0} = f^{abc} \neq 0$. Only for $N = 2$ it can be promoted to the totally antisymmetric $f^{abcd} = \epsilon^{abcd}$ without violating the "fundamental identity" (7.15). Fortunately, as mentioned in [7], the antisymmetry condition depends on the metric h_{ab}, which does not appear in supersymmetry transformations and does not affect the closure of the algebra.

A way from Lie algebra structure constants to generic 3-algebras lies through solving a *linear* equation for a *linear* operator t_{AB} [4, 7, 8], $t_{AB}(P) = [A, B, P]$:

$$t_{AB}([C, D]) - t_{CD}([A, B]) = [C, t_{AB}(D)] - [D, t_{AB}(C)] \tag{7.24}$$

which should afterwards be constrained by a non-linear equation

$$t_{AB}\Big(t_{CD}(E)\Big) - t_{CD}\Big(t_{AB}(E)\Big) = t_{t_{AB}(C),D}(E) + t_{C,t_{AB}(D)}(E) \tag{7.25}$$

8. The simplified action (5.1) in the general BL framework

The $\mathrm{U}(N)$ Lie algebra decomposes as

$$\mathrm{U}(N) = \mathrm{U}(1) + \mathrm{SO}(N) + S(N) \tag{8.1}$$

(anti-Hermitian matrix is a sum of imaginary unit, real antisymmetric and imaginary symmetric) and this is a *symmetric* decomposition: $S(N)$ is not a subalgebra of $\mathrm{U}(N)$, only a representation of $\mathrm{SO}(N)$, but with the special property

$$[S(N), S(N)] \subset \mathrm{SO}(N) \tag{8.2}$$

– this is important for the algebraic discussion of [4]. It is also important that in $\mathrm{U}(N)$ one can multiply matrices, not only commute (there are d^{abc} constants in addition to f^{abc}). However, this possibility is actually not used in construction of Nambu bracket (1.1) [12, 7]

$$[A, B, C] = \mathrm{tr}(A) \cdot [B, C] + \mathrm{tr}(B) \cdot [C, A] + \mathrm{tr}(C) \cdot [A, B] \tag{8.3}$$

As already mentioned in the previous sections, there is no term with I (with non-vanishing trace) at the r.h.s. – this makes the bracket a kind of degenerate and f^{abcd} very asymmetric. In the case of $\mathrm{U}(2)$ one can add $I \cdot \mathrm{tr}([A, B]C)$, so that $[A, B, C]^d = \epsilon^{abcd} A_a B_b C_c$. It is easy to see that the linear operator

$$t_{AB}(*) = \mathrm{tr}(A) \cdot [B, *] - \mathrm{tr}(B) \cdot [A, *] + [A, B] \cdot \mathrm{tr}(*) \tag{8.4}$$

which defines (8.3), satisfies equations (7.24) and (7.25).

It remains to rewrite the action (7.14) in terms of this 3-product (taking care of the lack of total antisymmetry). We denote the extra 0-components (traces) of the fields $\phi^I_{i\bar{j}}$ and $\psi^A_{i\bar{j}}$ through φ^I and χ^A, while the BL field $\mathcal{A}_{ab} = \mathcal{A}_{i\bar{j},k\bar{l}} = \mathcal{A}^{jl}_{ik}$ is split into $\mathcal{A}^{jk}_{ik} = \frac{1}{2} A^j_i$ and $\mathcal{A}^{jk}_{ki} - \mathcal{A}^{kj}_{ik} = B^j_i$ (this is not a one-to-one change, as already mentioned in the end of section 7.2). With these notations we get, for example, from (7.9):

$$\tilde{\mathcal{A}}^{jp}_{iq} \phi^q_p = \mathcal{A}^{kj}_{kl} \phi^l_i - \mathcal{A}^{kl}_{ki} \phi^j_l - \mathcal{A}^{jk}_{lk} \phi^l_i + \mathcal{A}^{lk}_{ik} \phi^j_l + \phi^l_l \left(\mathcal{A}^{jk}_{ki} - \mathcal{A}^{kj}_{ik} \right) = [A, \phi]^j_i + B^j_i \varphi \tag{8.5}$$

Expressions (5.2) for long derivatives are immediate corollaries of this relation. In the same way one can deduce from (7.14) all other terms of the action (5.1).

9. Conclusion

To summarize, eq. (5.1) explicitly describes a presumably-$O\,\mathrm{Sp}(8|4)$-invariant BL world-volume action for an arbitrary number N of coincident $M2$ branes. Despite being based on a very primitive 3-algebra, associated with the $\mathrm{U}(N)$ Nambu bracket (1.1) [12, 7] and thus lacking the mysterious non-adjoint gauge fields A^μ_{ab}, this simplified action is sufficient to describe transition *a la* [8] to generic $D2$ branes with arbitrary gauge group. Thus it can provide a far simpler and down-to-Earth solution to the problem posed in [1] than the generic BL construction, deeply hiding the underlying 3-algebra structure. As explained in section 7.3 above, this is rather a drawback than an advantage of eq. (5.1).

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